THE BELL SYSTEM TECHNICAL JOURNAL

DEVOTED TO THE SCIENTIFIC AND ENGINEERING ASPECTS OF ELECTRICAL COMMUNICATION

Volume 57 | November 1978 | Number 9

Calculation of Steady-State Probabilities for Content of Buffer with Correlated Inputs

By W. A. MASSEY and J. A. MORRISON

(Manuscript received March 2, 1978)

In a previous paper, a model for the behavior of a switching node that receives data from many terminals over low-speed access lines was considered. In this paper, we give the details of an alternate procedure for calculating the steady-state probabilities for the buffer content. It is shown that a finite system of linear equations may be obtained for calculating the steady-state probability that the buffer content is i. In a particular case of interest, explicit formulas are derived for the number of equations which arise in this procedure, for each value of i. Some detailed calculations are given for one example.

I. INTRODUCTION

Mathematical models for the behavior of a switching node that receives data from a (large) number of terminals over low-speed access lines have been considered by Gopinath and Morrison,[1,2] and some particular examples have been investigated by Fraser, Gopinath, and Morrison.[3] In this paper, we consider one of the models and give the details of an alternate procedure, which was alluded to by Gopinath and Morrison,[1] for calculating certain steady-state probabilities.

We first describe the model which we will consider. It is assumed that the data are received at the switching node in the form of packets of fixed size. As the packets arrive, they are placed in a buffer, which is a first-in-first-out queue. The buffer processes packets at a uniform rate, provided that it is not empty. In an actual computer network, the buffer capacity is finite, and a packet is lost if the buffer is full when it attempts to enter it. In our mathematical model, it is assumed that the buffer has

infinite capacity, so that no overflow is possible, and we are interested in calculating the steady-state probability that the buffer content (i.e., the number of packets in the buffer) exceeds the proposed capacity of the buffer.

We let the time that it takes for the buffer to process a packet through the node be our unit of time. We suppose that ξ_n is the number of packets which enter the buffer in the time interval $(n,n+1]$. If b_n denotes the buffer content at time n, then the buffer content at time $n+1$ is given by the equation

$$b_{n+1} = (b_n - 1)^+ + \xi_n, \tag{1}$$

where $a^+ = \max(a,0)$. The quantity ξ_n is a random variable, and hence so is b_n.

Consider the case in which each message from a terminal consists of exactly two packets which are separated by k units of time, where k is an integer. The packets are spread apart since the speed of the access lines is slower than the buffer processing rate. If x_n denotes the number of first packets entering the buffer in the interval $(n,n+1]$, then $\xi_n = x_n + x_{n-k}$, since x_{n-k} is the number of second packets entering in this interval which belong to messages whose first packets entered k intervals earlier. It was shown,[1,3] under suitable conditions, that if the number of terminals is large, then it is a reasonable approximation to assume that the random variables x_i are independently and identically distributed (i.i.d.).

A generalization of the above model was considered,[1] in which the number of packets entering the buffer in the interval $(n,n+1]$ is

$$\xi_n = \sum_{j=0}^{k} \alpha_j x_{n-j}, \tag{2}$$

where the nonnegative integer valued random variables x_i are i.i.d. and the constant coefficients α_i are nonnegative integers. It is assumed, without loss of generality, that $\alpha_0 \neq 0 \neq \alpha_k$. This is the model which we consider in this paper. It corresponds to a fixed pattern for each message. A more general model was considered[2] which allows for randomness in the message pattern, e.g., a random number of packets in a message. It would be of interest to obtain results for the more general model, analogous to those derived in this paper for the model corresponding to (2). This could be the topic of a future paper.

The results are stated and proved in a series of propositions, lemmas, theorems, and corollaries. In Section II, an explicit expression is first given for the steady-state probability that the buffer is empty, under the assumption that the mean arrival rate is less than unity. The steady-state probability that the buffer content is i is expressed in terms of the steady-state probabilities corresponding to a certain $(k+1)$-dimensional Markov process. Criteria for the proper states of this process

are obtained, and it is shown that, for fixed k, a finite number of these states correspond to a prescribed buffer content. The fundamental relation satisfied by the steady-state probabilities corresponding to the $(k+1)$-dimensional process is derived.

In Section III, it is shown how this fundamental relation may be iterated, so as to obtain a finite system of linear equations for calculating the steady-state probabilities corresponding to a prescribed buffer content. Numerous subsidiary quantities are defined to establish the required reduction formulas. The use of the reduction formulas to obtain the desired steady-state probabilities is described in Section IV.

In Section V, attention is turned to the particular case $\xi_n = x_n + x_{n-k}$, so that $\alpha_0 = 1 = \alpha_k$, and $\alpha_j = 0$ otherwise, in (2). Explicit formulas are derived for the number of equations which occur in the calculation of the steady-state probabilities corresponding to a prescribed buffer content. In Section VI, the steady-state probabilities corresponding to an empty buffer are calculated in the case $k = 4$.

II. THE FUNDAMENTAL RELATION

We assume that the mean arrival rate at the buffer is less than unity, and we are interested in determining the quantities

$$\kappa_i = \lim_{n \to \infty} \Pr(b_n = i), \tag{3}$$

where b_n satisfies (1) subject to (2). Hence, κ_i is the steady-state probability that the buffer content is i. We will see that the determination of these quantities involves the determination of certain other steady-state probabilities, as discussed by Gopinath and Morrison.[1] It was proved[2] that all these steady-state probabilities exist. We proceed to state, and prove, the results in a series of propositions, lemmas, theorems, and corollaries. We first give an explicit expression[1] for κ_0, the steady-state probability that the buffer is empty.

Proposition 1: $\kappa_0 = 1 - \mu_k E(x)$ *where* $\mu_k = \sum_{i=0}^{k} \alpha_i$ *and* $E(x)$ *is the expectation of any* x_n.

(This result may be derived by solving (77) in Ref. 2 for the marginal generating function $\phi_k(s)$, and letting $s \to 1$. This was the method of proof used in Ref. 1.)

We remark that, from (2), $E(\xi_n) = \mu_k E(x)$. Note that our assumption that the mean arrival rate is less than unity implies that $\kappa_0 > 0$.

To determine the other κ_i's, it will be convenient to use the following quantities:

$$\theta_n^{(r)} = \sum_{i=r}^{k} \alpha_i x_{n+r-i-1} \qquad \text{for } r = 1, \ldots, k. \tag{4}$$

Since we are using the first packets of a message to count the number

of intermediate packets at some later time, the $\theta_n^{(r)}$ correspond in this sense to the packet contribution prior to time n to ξ_{n+r-1}. We will determine the κ_i's by exploiting the recursive relations between b_n, ξ_n, and $\theta_n^{(r)}$.

Let Z^I be the direct sum of a countable number of copies of Z, the set of integers. For l, a nonnegative integer, we define the following collection of subsets:

$$N^l = \{(n_0, \ldots, n_l, 0, \ldots) | n_0, \ldots, n_l \geq 0\}.$$

Clearly, we have $N^0 \subset N^1 \subset \ldots \subset Z^I$. We now define a random $k + 1$-tuple variable

$$\mathbf{B}_n = (b_n, \theta_n^{(1)}, \ldots, \theta_n^{(k)}, 0, \ldots). \tag{5}$$

Using $\mathbf{B}_n$, we can define a map U that sends Z^I into $[0,1]$. Given $\mathbf{m} \in Z^I$, with $\mathbf{m} = (m_0, m_1, \ldots)$, we define

$$U(\mathbf{m}) = \lim_{n \to \infty} Pr(\mathbf{B}_n = \mathbf{m}). \tag{6}$$

We can then recover any κ_i from the $U(\mathbf{m})$'s via the relation

$$\kappa_i = \sum_{m_0 = i} U(\mathbf{m}). \tag{7}$$

This summation looks unwieldy, but we will show that this is not the case.

We first establish

Proposition 2: $(b_{n-l} - 1)^+ + \sum_{i=1}^{l} \xi_{n-i} \leq b_n + l - 1$ *for* $l \geq 1$.

Proof: Use induction on l.

$(l = 1)$ $\quad (b_{n-1} - 1)^+ + \xi_{n-1} = b_n$, from (1).

$(l \to l + 1)$ $\quad$ Note that $(b_{n-l} - 1) \leq (b_{n-l} - 1)^+$, and hence, using (1),

$$(b_{n-l-1} - 1)^+ + \sum_{i=1}^{l+1} \xi_{n-i} = b_{n-l} + \sum_{i=1}^{l} \xi_{n-i}$$
$$\leq (b_{n-l} - 1)^+ + 1 + \sum_{i=1}^{l} \xi_{n-i} \leq b_n + l \;{**}$$

(The double asterisk is used throughout the paper to denote the end of a proof.)

We now prove

Theorem 3: $U(\mathbf{m}) \neq 0$ *implies that* $\mathbf{m} \in N^k$, α_k *divides* m_k *and, for* $l = 1, \ldots, k$,

$$\sum_{i=1}^{l} m_{k-i+1} \leq \sum_{j=1}^{l} \alpha_0^{-1} \alpha_{k-l+j} (m_0 + j - 1).$$

Proof: To have a nonzero probability that $\mathbf{B}_n = \mathbf{m}$, it is immediate that

$\mathbf{m} \in N^k$. Also, $\theta_n^{(k)} = \alpha_k x_{n-1} = m_k$, and for a nonzero probability, x_{n-1} must take on an integer value.

Using Proposition 2, we have for $l = 1, \ldots, k$,

$$b_n + l - 1 \geq \sum_{i=1}^{l} \xi_{n-i} = \sum_{i=1}^{l} \sum_{j=0}^{k} \alpha_j x_{n-i-j} \geq \alpha_0 \sum_{i=1}^{l} x_{n-i}.$$

Therefore,

$$\sum_{i=1}^{l} x_{n-i} \leq \alpha_0^{-1}(b_n + l - 1). \tag{8}$$

Now

$$\sum_{i=1}^{l} \theta_n^{(k-i+1)} = \sum_{i=1}^{l} \sum_{j=k-i+1}^{k} \alpha_j x_{n+k-i-j}$$
$$= \sum_{j=k-l+1}^{k} \sum_{i=k+1-j}^{l} \alpha_j x_{n+k-i-j}.$$

Hence, if we make the substitutions $j = \tau + k - l$ and $i = \sigma + l - \tau$, we obtain

$$\sum_{i=1}^{l} \theta_n^{(k-i+1)} = \sum_{\tau=1}^{l} \alpha_{\tau+k-l} \left(\sum_{\sigma=1}^{\tau} x_{n-\sigma} \right)$$
$$\leq \sum_{\tau=1}^{l} \alpha_0^{-1} \alpha_{\tau+k-l}(b_n + \tau - 1),$$

using (8).

So, if $\theta_n^{(r)} = m_r$ and $b_n = m_0$, the m_r's must satisfy these conditions. **

Corollary 4: For fixed m_0 and k, there can only be a finite number of $\mathbf{m}$ *such that $U(\mathbf{m}) \neq 0$.*

Such $\mathbf{m}$ that satisfy the criteria of Theorem 3 will be called proper states. From (7), each κ_i then is the sum over only a finite number of these.

To derive the fundamental relation satisfied by $U(\mathbf{m})$ we need

Proposition 5: $\theta_{n+1}^{(r)} = \alpha_r x_n + \theta_n{}^{(r+1)}$ for $r = 1, \ldots, k-1$ and $\theta_{n+1}^{(k)} = \alpha_k x_n$.

Proof: From (4), for $r = 1, \ldots, k-1$,

$$\theta_{n+1}^{(r)} = \sum_{i=r}^{k} \alpha_i x_{n+r-i}$$
$$= \alpha_r x_n + \sum_{i=r+1}^{k} \alpha_i x_{n+r+1-i-1}$$
$$= \alpha_r x_n + \theta_n^{(r+1)},$$

and $\theta_{n+1}^{(k)} = \alpha_k x_n$ by definition. **

We now define a map from Z^l into itself called T_γ, where γ is a nonnegative integer:

$$T_\gamma(\mathrm{m}) = R(\mathrm{m}) + (\gamma, -(\gamma - 1)^+, 0, \ldots),$$

where R is the right shift operator. More explicitly, we can write

$$T_\gamma(\mathbf{m}) = (\gamma, m_0 - (\gamma - 1)^+, m_1, \ldots). \tag{9}$$

Theorem 6:

$$U(\mathbf{m}) = p(\sigma) \textstyle\sum_{\gamma \geq 0} U(T_\gamma(\mathrm{m} - \sigma\nu_0)),$$

where $\sigma = \alpha_k^{-1} m_k, p(\sigma) = Pr(x_n = \sigma)$ *and* $\nu_0 = (\alpha_0, \ldots, \alpha_k, 0, \ldots)$.

Proof: By Proposition 5, $\theta_{n+1}^{(k)} = m_k$ implies $x_n = \alpha_k^{-1} m_k = \sigma$. If σ is not a nonnegative integer, then $p(\sigma) = 0$ and $U(\mathrm{m}) = 0$, from (5) and (6), and the equation holds trivially.

Now we let σ be a nonnegative integer. Recall from Proposition 5 again that

$$\theta_n^{(r+1)} = \theta_{n+1}^{(r)} - \alpha_r x_n, \qquad \text{for } r = 1, \ldots, k-1.$$

Also, from (1), (2) and (4), we have

$$\theta_n^{(1)} = \xi_n - \alpha_0 x_n = b_{n+1} - (b_n - 1)^+ - \alpha_0 x_n.$$

So, if $b_n = \gamma$ and $\mathbf{B}_{n+1} = (m_0, \ldots, m_k, 0, \ldots)$, it will be necessary and sufficient, from (5), that $x_n = \sigma$ and

$$\mathbf{B}_n = (\gamma, m_0 - (\gamma - 1)^+ - \alpha_0\sigma, m_1 - \alpha_1\sigma, \ldots, m_{k-1} - \alpha_{k-1}\sigma, 0, \ldots).$$

In more compact notation, for $\mathbf{m} \in N^k$ we have $\mathbf{B}_{n+1} = \mathbf{m}$, $b_n = \gamma$ iff $\mathbf{B}_n = T_\gamma(\mathrm{m} - \sigma\nu_0)$, $x_n = \sigma$.

But x_n is independent of b_n and hence, from (4) and (5), of $\mathbf{B}_n$. Therefore,

$$\begin{aligned}\Pr(\mathbf{B}_{n+1} = \mathbf{m}) &= \sum_{\gamma \geq 0} \Pr(\mathbf{B}_{n+1} = \mathbf{m}, b_n = \gamma) \\ &= \Pr(x_n = \sigma) \sum_{\gamma \geq 0} \Pr(\mathbf{B}_n = T_\gamma(\mathbf{m} - \sigma\nu_0)).\end{aligned}$$

The theorem follows by letting $n \to \infty$ and using (6). ..

The fundamental relation in Theorem 6 satisifed by $U(\mathbf{m})$ was stated by Gopinath and Morrison,[1] in less compact notation. They also showed[1,2] that, once the steady-state probabilities $U(\mathrm{m})$ with $m_0 = 0$, corresponding to an empty buffer, were obtained, then the steady-state generating function for the buffer content could be calculated in terms of the generating functions for some marginal distributions. In this paper, we show how the quantities $U(\mathbf{m})$ may be calculated for any value of m_0, so that the steady-state probability that the buffer content is i may be calculated with the help of (7). In fact, we show how to iterate the fundamental relation in Theorem 6 so as to obtain a finite system of linear equations for calculating $U(\mathrm{m})$ for a fixed value of m_0. This procedure was alluded to by Gopinath and Morrison[1] in the case $m_0 = 0$.

III. REDUCTION FORMULAS

We first remark that, if $m_k = 0$ then the summation in the fundamental relation for $U(\mathrm{m})$ in Theorem 6 includes the term corresponding

to $\gamma = m_0 + 1$, so that this does not give a closed system of equations for $U(\mathbf{m})$ for a given value of m_0. To carry out the desired iteration of the fundamental relation, it is convenient to define some new quantities. Accordingly, we let

$$U^{(1)}(\mathbf{m}) = \sum_{\gamma \geq 0} U(T_\gamma(\mathbf{m})), \tag{10}$$

and, for $r = 1, \ldots, k-1$, define

$$U^{(r+1)}(\mathbf{m}) = \sum_{\gamma \geq 1} U^{(r)}(T_\gamma(\mathbf{m})). \tag{11}$$

Note that the summation starts at $\gamma = 0$ in (10), but at $\gamma = 1$ in (11). In terms of the definition in (10), Theorem 6 may be restated as

Theorem 6′:

$$U(\mathbf{m}) = p(\sigma) U^{(1)}(\mathbf{m} - \sigma \nu_0),$$

where $\sigma = \alpha_k^{-1} m_k$, $p(\sigma) = Pr(x_n = \sigma)$ *and* $\nu_0 = (\alpha_0, \ldots, \alpha_k, 0, \ldots)$.

The $U^{(r)}$'s are intimately related to the U's, and analogous statements can be made about them.

Theorem 7: $U^{(r)}(\mathbf{m}) \neq 0$ *implies, for* $k \geq 1$ *and* $r = 1, \ldots k$, *that* $\mathbf{m} \in N^{k-r}$ *and, for* $k \geq 2$ *and* $r = 1, \ldots, k-1$, *that* α_k *divides* m_{k-r} *and, for* $l = 1, \ldots, k-r$,

$$\sum_{i=1}^{l} m_{k-i-r+1} \leq \sum_{j=1}^{l} \alpha_0^{-1} \alpha_{k-l+j} (m_0 + r + j - 1).$$

Proof: Use induction on r.

($r = 1$) From (10), $U^{(1)}(\mathbf{m}) \neq 0$ implies that $U(T_\gamma(\mathbf{m})) \neq 0$ for some $\gamma \geq 0$. By Theorem 3, $T_\gamma(\mathbf{m}) \in N^k$ and α_k divides $(T_\gamma(\mathbf{m}))_k$, for some $\gamma \geq 0$. Hence, from (9), $\mathbf{m} \in N^{k-1}$ and, for $k \geq 2$, α_k divides m_{k-1}. Using the inequalities in Theorem 3 on $T_\gamma(\mathbf{m})$, we have

$$\sum_{i=1}^{l} (T_\gamma(m))_{k-i+1} \leq \sum_{j=1}^{l} \alpha_0^{-1} \alpha_{k-l+j} (\gamma + j - 1),$$

for $l = 1, \ldots . k$. This translates into

$$\sum_{i=1}^{l} m_{k-i} \leq \sum_{j=1}^{l} \alpha_0^{-1} \alpha_{k-l+j} (\gamma + j - 1),$$

for $l = 1, \ldots, k-1$, and

$$\sum_{i=1}^{k} m_{k-i} - (\gamma - 1)^+ \leq \sum_{j=1}^{k} \alpha_0^{-1} \alpha_j (\gamma + j - 1).$$

Since $(T_\gamma(\mathbf{m}))_1 = m_0 - (\gamma - 1)^+$, we must have $\gamma \leq m_0 + 1$ in order for $T_\gamma(\mathbf{m}) \in N^k$. It is necessary that the m_i's satisfy the above inequalities for the largest possible γ, so we let $\gamma = m_0 + 1$. Then

$$\sum_{i=1}^{l} m_{k-i} \le \sum_{j=1}^{l} \alpha_0^{-1}\alpha_{k-l+j}(m_0 + j),$$

for $l = 1, \ldots, k - 1$, and the other inequality is redundant, being trivial for $k = 1$, and implied for $k \ge 2$ by the inequality for $l = k - 1$.

$(r \to r + 1)$ We consider $r \le k - 2$, for $k \ge 3$. From (11), $U^{(r+1)}(\mathbf{m}) \neq 0$ implies that $U^{(r)}(T_\gamma(\mathbf{m})) \neq 0$ for some $\gamma \ge 1$, so that $T_\gamma(\mathbf{m}) \in N^{k-r}$ and α_k divides $(T_\gamma(\mathrm{m}))_{k-r}$. Hence, from (9), $\mathbf{m} \in N^{k-r-1}$ and α_k divides m_{k-r-1}. Also, for some $\gamma \ge 1$,

$$\sum_{i=1}^{l} (T_\gamma(\mathbf{m}))_{k-i-r+1} \le \sum_{j=1}^{l} \alpha_0^{-1}\, \alpha_{k-l+j}(\gamma + r + j - 1),$$

for $l = 1, \ldots, k - r$. As before, $\gamma \le m_0 + 1$, and for $l = 1, \ldots k - r - 1$ we obtain the inequalities

$$\sum_{i=1}^{l} m_{k-i-r} \le \sum_{j=1}^{l} \alpha_0^{-1}\alpha_{k-l+j}(m_0 + r + j).$$

As before, the inequality for $l = k - r$ is redundant.

It follows from the above that $U^{(k-1)}(\mathbf{m}) \neq 0$ implies that $\mathbf{m} \in N^1$, for $k \ge 2$. Hence, from (11), $U^{(k)}(\mathbf{m}) \neq 0$, for $k \ge 2$, implies that $T_\gamma(\mathbf{m}) \in N^1$ for some $\gamma \ge 1$, so that, from (9), $\mathbf{m} \in N^0$. But we have already shown for $k = 1$ that $U^{(1)}(\mathbf{m}) \neq 0$ implies that $\mathrm{m} \in N^0$. Hence, $U^{(k)}(\mathrm{m}) \neq 0$ implies that $\mathbf{m} \in N^0$ for $k \ge 1$. ∗∗

Corollary 8: For fixed m_0,k,r, there is only a finite number of $\mathbf{m}$ *such that $U^{(r)}(\mathrm{m}) \neq 0$. Moreover, each sum that defines each $U^{(r)}$ is finite.*

Proof: The first assertion is clear. For the second, we use (10) and (11) and the fact that $(T_\gamma(\mathbf{m}))_1 = m_0 - (\gamma - 1).^+$ ∗∗

Before we derive the relations for $U^{(r)}(\mathbf{m})$ corresponding to Theorem 6′, we need some more definitions. For $r = 0, \ldots, k$ we define

$$\mu_r = \sum_{i=0}^{r} \alpha_i, \tag{12}$$

and, for $r = 0, \ldots, k - 1$,

$$\nu_r = (\mu_r, \alpha_{r+1}, \ldots, \alpha_k, 0, \ldots). \tag{13}$$

We will make use of

Proposition 9: T_γ has the properties:

(*i*) $T_\gamma(\mathbf{m} + \mathbf{m}') = T_\gamma(\mathbf{m}) + R(\mathbf{m}')$.

(*ii*) *For integers $\gamma \ge 1$ and $\gamma' \ge 0$,*

$$T_{\gamma+\gamma'}(\mathbf{m}) = T_\gamma(\mathrm{m}) + (\gamma', -\gamma', 0 \ldots).$$

(*iii*) *For integers $\gamma \ge 1$ and $\sigma \ge 0$, and $r = 1, \ldots, k - 2$,*

$$T_{\gamma+\sigma\mu_r}(\mathbf{m}) - \sigma\nu_r = T_\gamma(\mathbf{m} - \sigma\nu_{r+1}).$$

Proof: (i) and (ii) follow directly from (9). Also, using (ii), for integers $\gamma \geq 1$ and $\sigma \geq 0$, we have

$$\begin{aligned} T_{\gamma+\sigma\mu_r}(\mathbf{m}) - \sigma\nu_r &= T_\gamma(\mathbf{m}) + (\sigma\mu_r, -\sigma\mu_r, 0, \ldots) - \sigma\nu_r \\ &= T_\gamma(\mathbf{m}) - \sigma(0, \mu_r + \alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_k, 0, \ldots) \\ &= T_\gamma(\mathbf{m}) - R(\sigma\nu_{r+1}) = T_\gamma(\mathbf{m} - \sigma\nu_{r+1}). \;{**} \end{aligned}$$

For $k \geq 2$, $r = 0, \ldots, k-2$, and σ a nonnegative integer, we define

$$S_r(\mathbf{m};\sigma) = T_{\sigma\mu_r}(\mathbf{m}) - \sigma\nu_r. \tag{14}$$

Also, for $k \geq 2$, $r = 1, \ldots, k-1$ and $s = 1, \ldots, r$, and $\alpha_k^{-1}m_{k-r}$ a nonnegative integer, we define $V_r^{(s)}(\mathbf{m})$:

$$V_r^{(s)}(\mathbf{m}) = \sum_{\gamma_1, \ldots, \gamma_{r-s} \geq 1} U^{(s)}(S_{s-1}(T_{\gamma_1} \circ \ldots \circ T_{\gamma_{r-s}}(\mathbf{m}); \alpha_k^{-1}m_{k-r})),$$

if $s \neq r$, where $\circ$ denotes composition of the operators, and

$$V_r^{(r)}(\mathbf{m}) = U^{(r)}(S_{r-1}(\mathbf{m}; \alpha_k^{-1}m_{k-r})). \tag{15}$$

Lemma 10: $V_{r+1}^{(s)}(\mathbf{m}) = \sum_{\gamma \geq 1} V_r^{(s)}(T_\gamma(\mathbf{m}))$ *for* $k \geq 3, r = 1, \ldots, k-2$ *and* $s = 1, \ldots, r$, *and* $\alpha_k^{-1}\,(T_\gamma(\mathbf{m}))_{k-r}$ *a nonnegative integer.*

Proof: We will only consider the case $r \neq s$. The proof for $r = s$ requires only a slight modification. From (15),

$$\begin{aligned} &\sum_{\gamma \geq 1} V_r^{(s)}(T_\gamma(\mathbf{m})) \\ &= \sum_{\gamma \geq 1} \; \sum_{\gamma_1, \ldots, \gamma_{r-s} \geq 1} U^{(s)}(S_{s-1}(T_{\gamma_1} \circ \ldots \circ T_{\gamma_{r-s}} \circ T_\gamma(\mathbf{m}); \sigma)), \end{aligned}$$

where $\sigma = \alpha_k^{-1}(T_\gamma(\mathbf{m}))_{k-r}$. However, from (9), $(T_\gamma(\mathbf{m}))_{k-r} = m_{k-r-1}$ for $k - r \geq 2$. Therefore, $\sigma = \alpha_k^{-1}(T_\gamma(\mathbf{m}))_{k-r} = \alpha_k^{-1}(\mathbf{m})_{k-r-1}$, and if we let $\gamma = \gamma_{r+1-s}$, then the above expression is equal to $V_{r+1}^{(s)}(\mathbf{m})$. **

Theorem 11: *Let* $k \geq 2$. *For* $r = 1, \ldots, k-1$, *we have the following formulas:*

$$U^{(r)}(\mathbf{m}) = p(\sigma)\left[U^{(r+1)}(\mathbf{m} - \sigma\nu_r) + \sum_{s=1}^{r} V_r^{(s)}(\mathbf{m})\right]$$

where $\sigma = \alpha_k^{-1}(\mathbf{m})_{k-r}$ *and* $\sigma \neq 0$. *If* $\sigma = 0$, *then*

$$U^{(r)}(\mathbf{m}) = p(0)[U^{(r+1)}(\mathbf{m}) + V_r^{(1)}(\mathbf{m})].$$

Proof: We use induction on r.

($r = 1$) Note here that the two cases coincide. From (10) and Theorem 6′,

$$U^{(1)}(\mathbf{m}) = \sum_{\gamma \geq 0} U(T_\gamma(\mathbf{m})) = \sum_{\gamma \geq 0} p(\sigma)U^{(1)}(T_\gamma(\mathbf{m}) - \sigma\nu_0),$$

where $\sigma = \alpha_k^{-1}(T_\gamma(\mathbf{m}))_k$. But $(T_\gamma(\mathbf{m}))_k = m_{k-1}$ for $k \geq 2$; therefore, $\sigma = \alpha_k^{-1}m_{k-1}$ and so σ is independent of γ, and

$$U^{(1)}(\mathbf{m}) = p(\sigma) \sum_{\gamma \geq 0} U^{(1)}(T_\gamma(\mathbf{m}) - \sigma\nu_0).$$

If σ is not a nonnegative integer, then $p(\sigma) = 0$ and $U^{(1)}(\mathbf{m}) = 0$, and the required result holds trivially. If σ is a nonnegative integer, then we let $q = \gamma - \sigma\mu_0$ and obtain

$$U^{(1)}(\mathbf{m}) = p(\sigma) \sum_{q \geq -\sigma\mu_0} U^{(1)}(T_{q+\sigma\mu_0}(\mathbf{m}) - \sigma\nu_0).$$

The zeroth term of $T_{q+\sigma\mu_0}(\mathbf{m}) - \sigma\nu_0$ is q, so, by Theorem 7, any terms where $q < 0$ vanish. Hence,

$$\begin{aligned} U^{(1)}(\mathbf{m}) &= p(\sigma)\left[U^{(1)}(T_{\sigma\mu_0}(\mathbf{m}) - \sigma\nu_0) + \sum_{q \geq 1} U^{(1)}(T_{q+\sigma\mu_0}(\mathbf{m}) - \sigma\nu_0)\right] \\ &= p(\sigma)\left[U^{(1)}(S_0(\mathbf{m};\sigma)) + \sum_{q \geq 1} U^{(1)}(T_q(\mathbf{m} - \sigma\nu_1))\right] \\ &= p(\sigma)[V_1^{(1)}(\mathbf{m}) + U^{(2)}(\mathbf{m} - \sigma\nu_1)]. \end{aligned}$$

The last two steps follow from (11), (14), and (15), and Proposition 9, and the fact that $\sigma = \alpha_k^{-1} m_{k-1}$, to use the definition of $V_1^{(1)}$.

$(r \rightarrow r+1)$ We consider $r \leq k-2$, for $k \geq 3$, and first assume that $\sigma = \alpha_k^{-1} m_{k-r-1} \neq 0$. But $(T_\gamma(\mathbf{m}))_{k-r} = m_{k-r-1}$, for $r \leq k-2$. Hence, $\sigma = \alpha_k^{-1}(T_\gamma(\mathbf{m}))_{k-r} \neq 0$, and we may use our inductive hypothesis on $U^{(r)}(T_\gamma(\mathbf{m}))$. From (11), since σ is independent of γ, we obtain

$$U^{(r+1)}(\mathbf{m}) = p(\sigma) \sum_{\gamma \geq 1}\left[U^{(r+1)}(T_\gamma(\mathbf{m}) - \sigma\nu_r) + \sum_{s=1}^{r} V_r^{(s)}(T_\gamma(\mathbf{m}))\right].$$

If $\sigma \neq 0$ is not a positive integer, then $p(\sigma) = 0$ and $U^{(r+1)}(\mathbf{m}) = 0$, and the required result holds trivially.

If σ is positive, then, using Lemma 10, we have

$$U^{(r+1)}(\mathbf{m}) = p(\sigma)\left[\sum_{\gamma \geq 1} U^{(r+1)}(T_\gamma(\mathbf{m}) - \sigma\nu_r) + \sum_{s=1}^{r} V_{r+1}^{(s)}(\mathbf{m})\right].$$

Also, if we let $q = \gamma - \sigma\mu_r$, then

$$\sum_{\gamma \geq 1} U^{(r+1)}(T_\gamma(\mathbf{m}) - \sigma\nu_r) = \sum_{q \geq 1-\sigma\mu_r} U^{(r+1)}(T_{q+\sigma\mu_r}(\mathbf{m}) - \sigma\nu_r).$$

But σ and μ_r are positive integers, so $\sigma\mu_r \geq 1$. Hence, by a similar argument to the case $r = 1$,

$$\begin{aligned} \sum_{\gamma \geq 1} U^{(r+1)}(T_\gamma(\mathbf{m}) - \sigma\nu_r) &= \sum_{q \geq 0} U^{(r+1)}(T_{q+\sigma\mu_r}(\mathbf{m}) - \sigma\nu_r) \\ &= U^{(r+1)}(T_{\sigma\mu_r}(\mathbf{m}) - \sigma\nu_r) + \sum_{q \geq 1} U^{(r+1)}(T_q(\mathbf{m} - \sigma\nu_{r+1})) \\ &= V_{r+1}^{(r+1)}(\mathbf{m}) + U^{(r+2)}(\mathbf{m} - \sigma\nu_{r+1}), \end{aligned}$$

where we have used (11), (14), and (15), and Proposition 9. Conse-

quently,

$$U^{(r+1)}(\mathbf{m}) = p(\sigma)\left[U^{(r+2)}(\mathbf{m}-\sigma\nu_{r+1}) + \sum_{s=1}^{r+1} V_{r+1}^{(s)}(\mathbf{m})\right],$$

where $\sigma = \alpha_k^{-1} m_{k-r-1}$.

Finally, we consider the case $\sigma = 0$, so that $(T_\gamma(\mathbf{m}))_{k-r} = m_{k-r-1} = 0$. Then, using (11) and the inductive hypothesis on $U^{(r)}(T_\gamma(\mathbf{m}))$, we have

$$U^{(r+1)}(\mathbf{m}) = p(0) \sum_{\gamma \geq 1} [U^{(r+1)}(T_\gamma(\mathbf{m})) + V_r^{(1)}(T_\gamma(\mathbf{m}))]$$
$$= p(0)[U^{(r+2)}(\mathbf{m}) + V_{r+1}^{(1)}(\mathbf{m})],$$

from Lemma 10. $_{**}$

Having derived the reduction formulas of Theorem 11, we now comment on the quantities $V_r^{(s)}(m)$ defined in (15), under the assumption that $\alpha_k^{-1} m_{k-r}$ is a nonnegative integer. It may be verified, from the definitions in (9), (13), and (14), that, for $k \geq 2$ and $r = 1, \ldots, k-1, S_{r-1}(\mathbf{m}; \alpha_k^{-1} m_{k-r}) \in N^{k-r}$ implies that $\mathbf{m} \in N^{k-r}$. Also, for $k \geq 3$, $r = 2, \ldots, k-1, s = 1, \ldots, r-1$, and positive integers $\gamma_1, \ldots, \gamma_{r-s}$, $S_{s-1}(T_{\gamma_1} \circ \ldots \circ T_{\gamma_{r-s}}(\mathbf{m});\ \alpha_k^{-1} m_{k-r}) \in \mathrm{N}^{k-s}$ implies that $\mathbf{m} \in N^{k-r}$. It follows, from Theorem 7, that, for $k \geq 2$, $r = 1, \ldots, k-1$ and $s = 1, \ldots, r$, $V_r^{(s)}(\mathbf{m}) \neq 0$ implies that $\mathbf{m} \in N^{k-r}$. Also, for $r = 0, \ldots, k-1$, we note that $(\mathbf{m} - \sigma\nu_r) \in N^{k-r-1}$, where $\sigma = \alpha_k^{-1} m_{k-r}$ is a nonnegative integer, implies that $\mathbf{m} \in N^{k-r}$.

IV. THE STEADY-STATE PROBABILITIES

We define the sets

$$\Omega_i = \{U(\mathbf{m}) | 0 \leq m_0 \leq i, \mathbf{m} \text{ a proper state}\}, \tag{16}$$

where the proper states satisy the criteria of Theorem 3. The sets Ω_i are finite, for fixed k, by Corollary 4. We will first show how to calculate the elements of Ω_0. Then, as shown by Gopinath and Morrison,[1,2] the steady-state generating function for the buffer content can be calculated in terms of the generating functions for some marginal distributions. The marginals are finitely solvable, in the sense that a finite number of components of the marginal distributions can be solved for, from a finite number of linear equations. However, we will give an alternate method for calculating the steady-state probability that the buffer content is i, which also involves a finite number of linear equations. In fact, by induction on i, we show how to calculate the elements of Ω_i, $i = 1, 2, \ldots$, and hence $\kappa_1, \ldots, \kappa_i$, from (7).

We begin by defining the sets

$$\Lambda_i = \{U^{(r)}(\mathrm{m}) | 0 \leq m_0 \leq i, 1 \leq r \leq k, \mathbf{m} \text{ a proper state}\}, \tag{17}$$

where the proper states satisfy the criteria of Theorem 7. The sets Λ_i are finite, for fixed k, by Corollary 8. We also define the sets Λ_i^*, which are

obtained from Λ_i by deleting the single element $U^{(k)}(i,0,0,\ldots)$, that is

$$\Lambda_i^* = \Lambda_i \sim \{U^{(k)}(i,0,0,\ldots)\}. \tag{18}$$

We will first show how to determine the elements of Λ_0^*, and thence the elements of Ω_0. We will make use of

Lemma 12: *For $k \geq 2$, $r = 1,\ldots,k-1$ and $s = 1,\ldots r$, the quantities $V_r^{(s)}(\mathbf{m})$ are linear combinations of the elements of Λ_0^*.*
Proof: From (9), (13), and (14), it follows that $(S_{s-1}(\mathbf{m};\sigma))_0 = 0$. The result is then a consequence of the definitions in (15), (17), and (18). ∗∗

If $k=1$, then Λ_0 contains the single element $U^{(1)}(0,0,\ldots)$, since $\mathbf{m}\in N^{k-r}$ for a proper state, and hence the set Λ_0^* is empty. If $k \geq 2$, then Λ_0^* contains at least one element, namely, $U^{(k-1)}(0,0,\ldots)$. (If $k = 2$, this might be the only element.) We now apply the reduction formula of Theorem 11 to each element of $\Lambda_0^* \sim U^{(k-1)}(0,0,\ldots)$. But for $m_0 = 0$ and σ a positive integer, $(\mathbf{m} - \sigma\nu_r)_0 < 0$, and hence $U^{(r+1)}(\mathbf{m} - \sigma\nu_r) = 0$. Hence, from Lemma 12, we obtain a system of homogeneous linear equations which contain as unknowns only the elements of Λ_0^*. Note that we have omitted the reduction formula for $U^{(k-1)}(0,0,\ldots)$. Since there is one more unknown than the number of equations, we can solve for the elements of Λ_0^* to within a multiplicative constant.

We are now in a position to determine the elements of Ω_0. If $k = 1$, then, from the inequality in Theorem 3, Ω_0 contains just the single element $U(0,0,\ldots) = \kappa_0$, from (7), and κ_0 is given by the formula in Proposition 1. If $k \geq 2$, then the elements of Ω_0 are given by Theorem 6′ in terms of elements of Λ_0^*, since $(\mathbf{m} - \sigma\nu_0)_0 \leq 0$ if $m_0 = 0$ and σ is a nonnegative integer, and $U^{(1)}(\mathbf{m} - \sigma\nu_0) = 0$ if $(\mathbf{m} - \sigma\nu_0)_0 < 0$. Hence, the elements of Ω_0 are determined to within a multiplicative constant, which is determined by (7), in terms of κ_0. The elements of Λ_0^* are now also completely determined.

We next turn our attention to the calculation of the elements of Λ_i^* and Ω_i, for $i = 1,2,\ldots$. First, however, we need

Lemma 13: *The assumption $\mu_k E(x) < 1$ implies that $p(0) > 0$.*
Proof: $E(x) = \sum_{i=1}^{\infty} ip(i) \geq \sum_{i=1}^{\infty} p(i) = 1 - p(0)$. But, from (12), since $\alpha_0 \neq 0 \neq \alpha_k$, it follows that $\mu_k \geq 2$, and hence $E(x) < 1/2$. ∗∗

We have shown how to determine the elements of Λ_0^* and Ω_0. We will show how to determine the elements of Λ_i^* and Ω_i, for $i = 1,2,\ldots$. We first consider the special case $k=1$, and use induction on i.

Theorem 14: *For $k = 1$, if the elements of Λ_i^* and Ω_i are known, then the elements of $\Lambda_{i+1}^* \sim \Lambda_i^*$ and $\Omega_{i+1} \sim \Omega_i$ may be determined.*
Proof: From (17) and (18), since $k = 1$ and $\mathbf{m}\in N^{k-r}$ for a proper state,

$$\Lambda_{i+1}^* \sim \Lambda_i^* = \{U^{(1)}(i,0,\ldots)\}.$$

But, from Theorem 6′,

$$U(i,0,\ldots) = p(0)U^{(1)}(i,0,\ldots).$$

Since $p(0) > 0$, by Lemma 13, and $U(i,0,\ldots) \in \Omega_i$, this equation determines $U^{(1)}(i,0,\ldots)$. Also, from Theorem 6′, $U(i+1,m_1,0,\ldots)$ is determined for $m_1 \neq 0$, if there are any such elements in Ω_{i+1}, since $\sigma > 0$ and so $(\mathbf{m} - \sigma\nu_0)_0 < i+1$. The remaining element of $\Omega_{i+1} \sim \Omega_i$ is $U(i+1,0,\ldots)$, since $\mathbf{m} \in N^1$ for a proper state. But, from (9) and (10),

$$U(i+1,0,\ldots) = U^{(1)}(i,0,\ldots) - \sum_{\gamma=0}^{i} U(\gamma,i-(\gamma-1)^+,0,\ldots),$$

so that the remaining element is determined. $_{**}$

We now consider the general case, and establish

Theorem 15: *For $k \geq 2$, if the elements of Λ_i^* are known, then the elements of $\Lambda_{i+1}^* \sim \Lambda_i^*$ may be determined.*

Proof: From Theorem 11,

$$U^{(k-1)}(i,0,\ldots) = p(0)[U^{(k)}(i,0,\ldots) + V_{k-1}^{(1)}(i,0,\ldots)],$$

which equation was omitted for $i = 0$. This equation determines $U^{(k)}(i,0,\ldots)$, by Lemmas 12 and 13, since $U^{(k-1)}(i,0,\ldots) \in \Lambda_i^*$. Also, if $m_0 = i+1$, $1 \leq r \leq k-1$ and $\sigma = \alpha_k^{-1}m_{k-r} > 0$, then $U^{(r)}(\mathbf{m})$ is determined by Theorem 11, since $(\mathbf{m} - \sigma\nu_r)_0 < i+1$ for $\sigma > 0$. The remaining elements of $\Lambda_{i+1}^* \sim \Lambda_i^*$ are $U^{(r)}(\mathbf{m})$ with $m_0 = i+1$, $1 \leq r \leq k-1$ and $m_{k-r} = 0$.

But from (11),

$$U^{(k-1)}(i+1,0,\ldots) = U^{(k)}(i,0,\ldots) - \sum_{\gamma=1}^{i} U^{(k-1)}(\gamma,i-\gamma+1,0,\ldots),$$

where the summation is absent if $i = 0$. This determines $U^{(k-1)}(i+1,0,\ldots)$, and if $k = 2$ this is the only remaining element in $\Lambda_{i+1}^* \sim \Lambda_i^*$. If $k \geq 3$, there still remain $U^{(r)}(\mathbf{m})$ with $m_0 = i+1$, $1 \leq r \leq k-2$ and $m_{k-r} = 0$, and from Theorem 11,

$$U^{(r)}(\mathbf{m}) = p(0)[U^{(r+1)}(\mathbf{m}) + V_r^{(1)}(\mathbf{m})].$$

But we have just determined $U^{(k-1)}(i+1,0,\ldots)$, and so we know $U^{(k-1)}(i+1,m_1,\ldots)$ for $m_1 \geq 0$. Hence, from the above equation, by Lemma 12, we may determine $U^{(k-2)}(\mathbf{m})$ with $m_0 = i+1$, and $m_2 = 0$. We then know $U^{(k-2)}(\mathbf{m})$ with $m_0 = i+1$ and $m_2 \geq 0$. By iteration of the above equation, we may determine any remaining elements of $\Lambda_{i+1}^* \sim \Lambda_i^*$. $_{**}$

Lemma 16: *For $k \geq 2$, the elements of Ω_i are determined by elements of Λ_i^*, for $i = 1,2,\ldots$.*

Proof: The result follows from Theorem 6′. $_{**}$

We have shown that the elements of Λ_i^* and Ω_i, for $i = 1,2,\ldots$, may be determined explicitly, once the elements of Λ_0^*, and Ω_0, are known.

The determination of the elements of Λ_0^*, however, involves the solution of a homogeneous system of linear equations.

V. A PARTICULAR CASE

We now confine our attention to the particular case $\xi_n = x_n + x_{n-k}$, so that, from (2),

$$\alpha_0 = 1 = \alpha_k, \quad \alpha_j = 0 \text{ otherwise.} \tag{19}$$

We are interested in determining the number of proper states m of $U(\mathrm{m})$, and also of $U^{(r)}(\mathrm{m})$, as defined by the criteria of Theorems 3 and 7. We show in the appendix that these criteria lead to a precise count of the number of nonzero U's and $U^{(r)}$'s when (19) holds, if $p(i) > 0$, $i = 0,1,2,\ldots$.

We will make use of

Lemma 17: *For* $r = -1,0,1,\ldots,$ *and* $s = 1,2,\ldots,$ *the number of elements of* $\mathbf{n} \in N^{s-1}$ *which satisfy the conditions* $\sum_{i=0}^{l-1} n_i \le r + l$ *for* $l = 1,\ldots,s$ *is*

$$P(r,s) = \binom{r+2s}{s} - \binom{r+2s}{s-2} = \frac{(r+2)(r+2s+1)!}{s!(r+s+2)!} \equiv F(r,s). \tag{20}$$

Proof: We use induction on s.

($s = 1$) The number of n_0 with $0 \le n_0 \le r+1$ is clearly $r + 2 = F(r,1)$.

($s \to s+1$) Now $\sum_{i=0}^{l-1} n_i \le r + l$ for $l = 1,\ldots,s+1$ implies that $n_0 \le r+1$ and $\sum_{i=1}^{l} n_i \le r + l + 1 - n_0$ for $l = 1,\ldots,s$. Hence,

$$P(r,s+1) = \sum_{n_0=0}^{r+1} P(r+1-n_0,s) = \sum_{i=0}^{r+1} P(i,s) = \sum_{i=0}^{r+1} F(i,s),$$

from the inductive hypothesis. But, as may be verified,

$$F(i,s) = F(i-1,s+1) - F(i-2,s+1). \tag{21}$$

Hence,

$$\sum_{i=0}^{r+1} F(i,s) = F(r,s+1), \tag{22}$$

since $F(-2,s+1) = 0$. **

Corollary 18: *For fixed* m_0 *and* k, *the number of proper states* m *of* $U(\mathrm{m})$ *is* $F(m_0-1,k)$, *and the number of proper states* $\mathbf{m}$ *of* $U^{(r)}(\mathrm{m})$ *is* $F(m_0 + r - 1, k - r)$, *for* $r = 1,\ldots,k$.

Proof: The results follow from (19), Theorems 3 and 7, and Lemma 17. Note that, for $r = k$, the only proper state of $U^{(k)}(\mathbf{m})$ is $(m_0,0,\ldots)$, since $\mathrm{m} \in N^0$, and we have $\mathrm{F}(m_0 + k - 1,0) = 1$. **

From (16) and Corollary 18, it follows that the number of elements of Ω_0 is

$$|\Omega_0| = F(-1,k) = \frac{(2k)!}{k!(k+1)!}. \tag{23}$$

Also, the number of elements of $\Omega_i \sim \Omega_0$ is

$$\begin{aligned}|\Omega_i| - |\Omega_0| &= \sum_{m_0=1}^{i} F(m_0 - 1,k) = F(i-2,k+1) \\ &= \frac{i(i+2k+1)!}{(k+1)!(i+k+1)!},\end{aligned} \tag{24}$$

from (20) and (22). From (17) and Corollary 18, the number of elements of Λ_i is

$$|\Lambda_i| = \sum_{m_0=0}^{i} \sum_{r=1}^{k} F(m_0 + r - 1,k - r). \tag{25}$$

But

$$F(-(r+4),r+s+2) = \frac{-(r+2)(r+2s+1)!}{(r+s+2)!s!} = -F(r,s). \tag{26}$$

Therefore, from (21) and (26), we have

$$F(m_0 + r - 1,k - r) = -F(-(m_0 + r + 3),m_0 + k + 1)$$

$$= -[F(-(m_0 + r + 4),m_0 + k + 2) - F(-(m_0 + r + 5),m_0 + k + 2)].$$

Hence, if we sum and use (26), we obtain

$$\sum_{r=1}^{k} F(m_0 + r - 1,k - r)$$

$$\begin{aligned}&= -[F(-(m_0 + 5),m_0 + k + 2) - F(-(m_0 + k + 5),m_0 + k + 2)] \\ &= F(m_0 + 1,k - 1) - F(m_0 + k + 1,-1) = F(m_0 + 1,k-1).\end{aligned} \tag{27}$$

From (21) and (25), it follows that

$$|\Lambda_i| = F(i,k) - F(-1,k). \tag{28}$$

Note, from (23), (24), and (28), that

$$|\Omega_i| + |\Lambda_i| = F(i-2,k+1) + F(i,k) = F(i-1,k+1),$$

from (21).

Of particular interest, for $k \geq 2$, is the number of equations required to determine the elements of Λ_0^* to within a multiplicative constant, namely $|\Lambda_0^*| - 1 = |\Lambda_0| - 2$, from (18). But, from (21) and (28),

$$|\Lambda_0| = F(0,k) - F(-1,k) = F(1,k-1) = \frac{3(2k)!}{(k-1)!(k+2)!}.$$

The first few values of $|\Lambda_0| - 2$ and $|\Omega_0|$, as given by (23), are

k	2	3	4	5	6
$\|\Lambda_0\| - 2$	1	7	26	88	295
$\|\Omega_0\|$	2	5	14	42	132

We also have the asymptotic result

$$\lim_{k\to\infty} \frac{(|\Lambda_0| - 2)}{|\Omega_0|} = 3.$$

VI. AN EXPLICIT EXAMPLE

We here consider the example corresponding to $k = 4$ in (19), so that $\xi_n = x_n + x_{n-4}$. We will explicitly determine the elements of Ω_0 for this example. As discussed in Section IV, the reduction formula of Theorem 11 is applied to each element of $\Lambda_0^* \sim U^{(3)}(0,0,\ldots)$. Then the elements of Ω_0 are determined with the help of Theorem 6′ and the normalization condition (7) with $i = 0$.

From (17), (18), and Theorem 7, the elements of Λ_0^*, with an obvious change of notation, are

$$U^{(1)}_{0m_1m_2m_3}, \quad m_3 \le 1, m_2 + m_3 \le 2, m_1 + m_2 + m_3 \le 3, \tag{29}$$

$$U^{(2)}_{0m_1m_2}, \quad m_2 \le 2, m_1 + m_2 \le 3, \tag{30}$$

and

$$U^{(3)}_{0m_1}, \quad m_1 \le 3, \tag{31}$$

where m_1, m_2 and m_3 are nonnegative integers. From (16) and Theorem 3, the elements of Ω_0 are

$$U_{0m_1m_2m_30}, \quad m_3 \le 1, m_2 + m_3 \le 2, m_1 + m_2 + m_3 \le 3. \tag{32}$$

But from Theorem 6′, again with an obvious change of notation,

$$U_{0m_1m_2m_30} = p_0 U^{(1)}_{0m_1m_2m_3}. \tag{33}$$

From (7), the normalization condition is

$$\kappa_0 = \sum U_{0m_1m_2m_30} = p_0 \sum U^{(1)}_{0m_1m_2m_3}, \tag{34}$$

where the summations are over the range of subscripts satisfying the inequalities in (29) and (32).

We now apply the reduction formula of Theorem 11 to each element of $\Lambda_0^* \sim U^{(3)}_{00}$, and note, from (12) and (19), that

$$\mu_r = 1, \qquad r = 0,1,2,3. \tag{35}$$

From (9), (13), and (14), with $\mathbf{m} = (m_0, m_1, m_2, m_3, 0, \ldots)$, we have

$$\mathbf{m} - m_3\nu_1 = (m_0 - m_3, m_1, m_2, 0, \ldots), \tag{36}$$

and

$$S_0(\mathbf{m};m_3) = (0,m_0 - (m_3 - 1)^+,m_1,m_2,0, \ldots). \tag{37}$$

Hence, from (15) and (37),

$$V_1^{(1)}(\mathbf{m}) = U^{(1)}_{0,m_0-(m_3-1)^+,m_1,m_2} \cdot \tag{38}$$

It follows from Theorem 11 that

$$U^{(1)}_{0m_1m_20} = p_0(U^{(2)}_{0m_1m_2} + U^{(1)}_{00m_1m_2}), \tag{39}$$

and

$$U^{(1)}_{0m_1m_21} = p_1 U^{(1)}_{00m_1m_2}, \tag{40}$$

since $U^{(2)}_{-1,m_1,m_2} = 0$.

Similarly, with $\mathbf{m} = (m_0,m_1,m_2,0, \ldots)$,

$$\mathbf{m} - m_2\nu_2 = (m_0 - m_2,m_1,0, \ldots), \tag{41}$$

$$S_1(\mathbf{m};m_2) = (0,m_0 - (m_2 - 1)^+,m_1,0, \ldots), \tag{42}$$

and

$$S_0(T_{\gamma_1}(\mathbf{m});m_2) = (0,\gamma_1 - (m_2 - 1)^+,m_0 - (\gamma_1 - 1)^+,m_1,0, \ldots). \tag{43}$$

Hence, from (15),

$$V_2^{(2)}(\mathbf{m}) = U^{(2)}_{0,m_0-(m_2-1)^+,m_1}, \tag{44}$$

and

$$V_2^{(1)}(\mathbf{m}) = \sum_{\gamma_1 \geq 1} U^{(1)}_{0,\gamma_1-(m_2-1)^+,m_0-(\gamma_1-1)^+,m_1}. \tag{45}$$

It follows from Theorem 11 that

$$U^{(2)}_{0m_10} = p_0(U^{(3)}_{0m_1} + U^{(1)}_{010m_1}), \tag{46}$$

and, for $m_2 \neq 0$,

$$U^{(2)}_{0m_1m_2} = p_{m_2}(U^{(2)}_{0,-(m_2-1)^+,m_1} + U^{(1)}_{0,1-(m_2-1)^+,0,m_1}). \tag{47}$$

Hence,

$$U^{(2)}_{0m_11} = p_1(U^{(2)}_{00m_1} + U^{(1)}_{010m_1}) \tag{48}$$

and

$$U^{(2)}_{0m_12} = p_2 U^{(1)}_{000m_1}. \tag{49}$$

Next, with $\mathbf{m} = (m_0,m_1,0, \ldots)$,

$$\mathbf{m} - m_1\nu_3 = (m_0 - m_1,0, \ldots), \tag{50}$$

$$S_2(\mathbf{m};m_1) = (0,m_0 - (m_1 - 1)^+,0, \ldots), \tag{51}$$

$$S_1(T_{\gamma_1}(\mathbf{m});m_1) = (0,\gamma_1 - (m_1 - 1)^+,m_0 - (\gamma_1 - 1)^+,0, \ldots), \tag{52}$$

and

$$S_0(T_{\gamma_1}(T_{\gamma_2}(\mathbf{m}));m_1)$$
$$= (0,\gamma_1 - (m_1 - 1)^+,\gamma_2 - (\gamma_1 - 1)^+, m_0 - (\gamma_2 - 1)^+,0,\ldots). \quad (53)$$

Hence, from (15),

$$V_3^{(3)}(\mathbf{m}) = U^{(3)}_{0,m_0-(m_1-1)^+}, \quad (54)$$

$$V_3^{(2)}(\mathbf{m}) = \sum_{\gamma_1\geq 1} U^{(2)}_{0,\gamma_1-(m_1-1)^+,m_0-(\gamma_1-1)^+}, \quad (55)$$

and

$$V_3^{(1)}(\mathbf{m}) = \sum_{\gamma_1,\gamma_2\geq 1} U^{(1)}_{0,\gamma_1-(m_1-1)^+,\gamma_2-(\gamma_1-1)^+,m_0-(\gamma_2-1)^+}. \quad (56)$$

It follows from Theorem 11 that, for $m_1 \neq 0$,

$$U^{(3)}_{0m_1} = p_{m_1}\Big(U^{(3)}_{0,-(m_1-1)^+} + U^{(2)}_{0,1-(m_1-1)^+,0} + \sum_{\gamma_1\geq 1} U^{(1)}_{0,\gamma_1-(m_1-1)^+,1-(\gamma_1-1)^+,0}\Big). \quad (57)$$

We now write out in full the nontrivial equations corresponding to (39), (40), (46), (48), (49), and (57), omitting terms which are identically zero. From (39) we have

$$U^{(1)}_{0000} = p_0(U^{(2)}_{000} + U^{(1)}_{0000}), \qquad U^{(1)}_{0010} = p_0(U^{(2)}_{001} + U^{(1)}_{0001}),$$
$$U^{(1)}_{0020} = p_0U^{(2)}_{002}, \qquad U^{(1)}_{0100} = p_0(U^{(2)}_{010} + U^{(1)}_{0010}),$$
$$U^{(1)}_{0110} = p_0(U^{(2)}_{011} + U^{(1)}_{0011}), \qquad U^{(1)}_{0120} = p_0U^{(2)}_{012},$$
$$U^{(1)}_{0200} = p_0(U^{(2)}_{020} + U^{(1)}_{0020}), \qquad U^{(1)}_{0210} = p_0U^{(2)}_{021},$$
$$U^{(1)}_{0300} = p_0U^{(2)}_{030}, \quad (58)$$

and from (40) we have

$$U^{(1)}_{0001} = p_1U^{(1)}_{0000}, \qquad U^{(1)}_{0011} = p_1U^{(1)}_{0001}, \qquad U^{(1)}_{0101} = p_1U^{(1)}_{0010},$$
$$U^{(1)}_{0111} = p_1U^{(1)}_{0011}, \qquad U^{(1)}_{0201} = p_1U^{(1)}_{0020}. \quad (59)$$

Next, from (46) we have

$$U^{(2)}_{000} = p_0(U^{(3)}_{00} + U^{(1)}_{0100}), \qquad U^{(2)}_{010} = p_0(U^{(3)}_{01} + U^{(1)}_{0101}),$$
$$U^{(2)}_{020} = p_0U^{(3)}_{02}, \qquad U^{(2)}_{030} = p_0U^{(3)}_{03}, \quad (60)$$

from (48) we have

$$U^{(2)}_{001} = p_1(U^{(2)}_{000} + U^{(1)}_{0100}),$$
$$U^{(2)}_{011} = p_1(U^{(2)}_{001} + U^{(1)}_{0101}), \qquad U^{(2)}_{021} = p_1U^{(2)}_{002}, \quad (61)$$

and from (49) we have

$$U^{(2)}_{002} = p_2U^{(1)}_{0000}, \qquad U^{(2)}_{012} = p_2U^{(1)}_{0001}. \quad (62)$$

Finally, from (57) we have

$$U_{01}^{(3)} = p_1(U_{00}^{(3)} + U_{010}^{(2)} + U_{0110}^{(1)} + U_{0200}^{(1)}),$$
$$U_{02}^{(3)} = p_2(U_{000}^{(2)} + U_{0010}^{(1)} + U_{0100}^{(1)}), \qquad U_{03}^{(3)} = p_3U_{0000}^{(1)}. \tag{63}$$

We may eliminate the 13 nonzero quantities $U_{0m_1m_2}^{(2)}$ and $U_{0m_1}^{(3)}$ from (58) to (63), and solve for the 14 nonzero quantities $U_{0m_1m_2m_3}^{(1)}$ to within a multiplicative constant. It is found that

$$U_{0000}^{(1)} = a_0, \qquad U_{0001}^{(1)} = p_1a_0, \qquad U_{0011}^{(1)} = p_1^2a_0,$$
$$U_{0111}^{(1)} = p_1^3a_0, \qquad U_{0020}^{(1)} = p_0p_2a_0, \qquad U_{0300}^{(1)} = p_0^2p_3a_0,$$
$$U_{0120}^{(1)} = U_{0210}^{(1)} = U_{0201}^{(1)} = p_0p_1p_2a_0, \tag{64}$$

and

$$U_{0010}^{(1)} = p_1\Delta a_0, \qquad U_{0101}^{(1)} = p_1^2\Delta a_0,$$
$$U_{0110}^{(1)} = p_1^2(1 + p_0p_1)\Delta a_0, \qquad U_{0200}^{(1)} = p_0p_2(1 + p_0p_1)\Delta a_0,$$
$$U_{0100}^{(1)} = p_1[1 + p_0^2(p_1^2 + p_0p_2)(1 + p_0p_1)]\Delta a_0, \tag{65}$$

where

$$\Delta = \{1 - p_0p_1[1 + p_0{}^2(p_1{}^2 + p_0p_2)(1 + p_0p_1)]\}^{-1}. \tag{66}$$

The constant a_0 is determined by the normalization condition (34). These results are consistent with those derived by a different method.[3]

APPENDIX

We show here that in the particular case corresponding to (19), the criteria of Theorems 3 and 7 lead to a precise count of the number of nonzero U's and $U^{(r)}$'s if $p(i) > 0$, $i = 0,1,2,\ldots$. We first prove

Theorem 19: If (19) *holds,* $p(i) > 0$, $i = 0,1,2,\ldots,$ $\mathbf{m} \in N^k$ *and*

$$\sum_{i=1}^{l} m_{k-i+1} \leq m_0 + l - 1, \qquad l = 1,\ldots,k, \tag{67}$$

then $U(\mathbf{m}) \neq 0$.
Proof: From (4), (5), and (19), it follows that

$$\mathbf{B}_n = (b_n, x_{n-k},\ldots,x_{n-1},0,\ldots). \tag{68}$$

It was shown[2] that the irreducible Markov chain, with state space consisting of those states which communicate with $(0,0,\ldots)$, is positive recurrent. Moreover, it was also shown that, in the present notation, the state $(i_0,0,\ldots)$ communicates with the state $(0,0,\ldots)$, where i_0 is a positive integer. Hence, with probability 1, the state $(i_0,0,\ldots)$ occurs

infinitely often. We now assume that $\mathbf{B}_{n-k}$ is given by

$$b_{n-k} = m_0 + k - \sum_{i=1}^{k} m_{k-i+1}, \qquad x_{n-2k+r-1} = 0, \qquad r = 1, \ldots ,k. \tag{69}$$

Also, with positive probability, since $p(i) > 0, \quad i = 0,1,2, \ldots,$

$$x_{n-k+r-1} = m_r, \qquad r = 1, \ldots ,k. \tag{70}$$

We will show that (67), (69), and (70) imply that $b_n = m_0$, and hence, from (6), that $U(\mathrm{m}) \neq 0$.

We first show, by induction, that

$$b_{n-k+r} = m_0 + k - r - \sum_{i=1}^{k-r} m_{k-i+1} \geq 1, \qquad r = 0, \ldots ,k-1. \tag{71}$$

This is true for $r = 0$, from (67) and (69).

$(r \rightarrow r + 1)$ We consider $r = 0, \ldots ,k - 2$, for $k \geq 2$. Since $\xi_n = x_n + x_{n-k}$, it follows from (1), (69), and (70) that

$$\begin{aligned} b_{n-k+r+1} &= (b_{n-k+r} - 1)^+ + m_{r+1} \\ &= b_{n-k+r} - 1 + m_{r+1} \\ &= m_0 + k - (r+1) - \sum_{i=1}^{k-r-1} m_{k-i+1} \geq 1, \end{aligned} \tag{72}$$

from (67). This completes the inductive proof of (71). Finally, with the help of (71), we obtain

$$b_n = (b_{n-1} - 1)^+ + m_k = b_{n-1} - 1 + m_k = m_0. \;{**}$$

We now prove

Theorem 20: Suppose that (19) *holds and* $p(i) > 0, \quad i = 0,1,2,\ldots.$ *Also suppose, for* $k \geq 1$ *and* $r = 1, \ldots ,k$, *that* $\mathbf{m} \in N^{k-r}$ *and, for* $k \geq 2$ *and* $r = 1, \ldots ,k - 1$, *that*

$$\sum_{i=1}^{l} m_{k-i-r+1} \leq m_0 + r + l - 1, \qquad l = 1, \ldots ,k - r. \tag{73}$$

Then $U^{(r)}(\mathrm{m}) \neq 0$.

Proof: Use induction on r. We note, from (6), (10), and (11), that $U^{(r)}(\mathbf{m}) \geq 0, \quad r = 1, \ldots ,k$.

$(r = 1)$ Let

$$\hat{\mathbf{m}} = T_{m_0+1}(\mathbf{m}) = (m_0 + 1, 0, m_1, \ldots), \tag{74}$$

from (9). We will show that $U(\hat{\mathbf{m}}) \neq 0$, which implies that $U^{(1)}(\mathrm{m}) \neq 0$, from (10). If $k = 1$, then $\hat{\mathbf{m}} = (m_0 + 1,0,0, \ldots)$, hence $\hat{\mathbf{m}} \in N^1$ and $0 = \hat{m}_1 \leq \hat{m}_0 = m_0 + 1$. It follows from Theorem 19 that $U(\hat{\mathbf{m}}) \neq 0$. If $k \geq 2$, then $\hat{\mathrm{m}} \in N^k$ since $\mathbf{m} \in N^{k-1}$, and, from (73),

$$\sum_{i=1}^{l} \hat{m}_{k-i+1} \le \hat{m}_0 + l - 1, \qquad l = 1, \ldots, k-1,$$

$$\sum_{i=1}^{k} \hat{m}_{k-i+1} = \sum_{i=1}^{k-1} \hat{m}_{k-i+1} \le \hat{m}_0 + k - 2 \le \hat{m}_0 + k - 1.$$

It follows from Theorem 19 that $U(\hat{\mathbf{m}}) \neq 0$.

$(r-1 \to r)$ We consider $r = 2, \ldots, k-1$, for $k \ge 3$. Then, from (74), $\mathbf{m} \in N^{k-r}$ implies that $\hat{\mathbf{m}} \in N^{k-r+1}$. Also, from (73),

$$\sum_{i=1}^{l} \hat{m}_{k-i-r+2} \le \hat{m}_0 + r + l - 2, \qquad l = 1, \ldots, k-r,$$

$$\sum_{i=1}^{k-r+1} \hat{m}_{k-i-r+2} = \sum_{i=1}^{k-r} \hat{m}_{k-i-r+2} \le \hat{m}_0 + k - 2 \le \hat{m}_0 + k - 1.$$

It follows from the inductive hypothesis that $U^{(r-1)}(\hat{\mathbf{m}}) \neq 0$. Hence, from (11), $U^{(r)}(\mathbf{m}) \neq 0$.

(k) $\mathbf{m} \in N^0$, for $k \ge 2$, implies that $\hat{\mathbf{m}} \in N^1$ and $0 = \hat{m}_1 \le \hat{m}_0 + k - 1 = m_0 + k$. Hence, $U^{(k-1)}(\hat{\mathbf{m}}) \neq 0$ and, from (11), $U^{(k)}(\mathbf{m}) \neq 0$. $**$

REFERENCES

1. B. Gopinath and J. A. Morrison, "A Discrete Queueing Problem Arising in Packet Switching," Analyse et Contrôle de Systèmes (1976), Séminaires IRIA, Rocquencourt, pp. 201–210.
2. B. Gopinath and J. A. Morrison, "Discrete-Time Single Server Queues with Correlated Inputs," B.S.T.J., *56*, No. 9 (November 1977), pp. 1743–1768.
3. A. G. Fraser, B. Gopinath, and J. A. Morrison, "Buffering of Slow Terminals," B.S.T.J., *57*, No. 8 (October 1978), pp. 2865–2885.